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Amenability and Virtual Diagonals for von Neumann Algebras

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It is shown that a certain universal bimodule over a von Neumann algebra is automatically normal. This leads to a simple direct proof of Haagerup's result that a von Neumann algebra is amenable, or equivalently, injective, if and only if it has a normal virtual diagonal. Haagerup has used this fact to prove that nuclear C^*-algebras are amenable. © 1988 Academic Press, Inc.

1. INTRODUCTION

A unital Banach algebra A is said to be *amenable* if all derivations of A into dual Banach A bimodules are inner (see below for definitions). Employing an elegant argument, B. E. Johnson showed that A is amenable if and only if it has a virtual diagonal [8, Lemma 1.2]. Much more recently, U. Haagerup has discovered a profound von Neumann algebraic analogue of this result [7] which uses normal virtual diagonals. This enabled him to prove that nuclear C^*-algebras are amenable (the converse had been proved by Connes 5 years earlier [2]).

Haagerup's proof that amenable von Neumann algebras have normal virtual diagonals does not at all resemble Johnson's simple demonstration for Banach algebras. This is due to the fact that it is unlikely that the the appropriate dual bimodules are normal. In this paper it is shown how an auxiliary normal dual bimodule suggested by Haagerup may be used to adapt Johnson's approach to von Neumann algebras.

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2. Some Tensor Product Bimodules

If V and W are Banach spaces, we let $\mathrm{Bil}(V, W)$ denote the bounded bilinear functions $F\colon V \times W \to \mathbb{C}$. This is a Banach space under the usual vector operations and the norm.

$$\|F\| = \sup\{|F(x, y)| : x \in V,\ y \in W,\ \|x\| = \|y\| = 1\}.$$

We shall identify the bilinear functions on $V \times W$ with the linear functions on the vector space tensor product $V \otimes W$, writing $f(x \otimes y) = f(x, y)$. The projective tensor product $V \hat{\otimes} W$ is the completion of $V \otimes W$ with respect to the norm

$$\|z\|^{\wedge} = \sup\{|F(z)| : F \in \mathrm{Bil}(V, W)_1\} \quad (z \in V \otimes W),$$

where $\mathrm{Bil}(V, W)_1$ is the closed unit ball of $\mathrm{Bil}(V, W)$. We may also define $\|\ \|^{\wedge}$ by

$$\|z\|^{\wedge} = \inf\left\{\sum_{i=1}^{n} \|x_i\|\,\|y_i\| : z = \sum x_i \otimes y_i\right\}$$

(see [3, Sect. IV.23]).

Given a unital Banach algebra A, a *normed* (A, B) *module* consists of a normal vector space V together with maps $(a, v) \to av$ and $(v, b) \to vb$ of $A \times V$ and $V \times B$ into V, satisfying the usual algebraic properties, and

$$\|av\| \leqslant \|a\|\,\|v\|, \qquad \|vb\| \leqslant \|v\|\,\|b\|.$$

Given two such (A, B) modules V and W, we say that a linear map $\theta\colon V \to W$ is an (A, B) *module map* if $\theta(av) = a\theta(v)$ and $\theta(vb) = \theta(v)\,b$ ($a \in A$, $b \in B$, $v \in V$). If $A = B$ we say that V is a normed A *bimodule*, and the corresponding maps are *bimodule maps*. The dual space V^* of a normed (A, B) *module* V is a normed (B, A) module under the operations

$$(bf)(a) = f(ab), (fa)(b) = f(ab).$$

We say that V^* is a *dual* normed (or Banach) (B, A) module.

Given Banach algebras A and B, $\mathrm{Bil}(A, B)$ is a Banach (B, A) module under the "external" operations

$$(bF)(x, y) = F(x, yb), \qquad (Fa)(x, y) = F(ax, y),$$

where $F \in \mathrm{Bil}(A, B)$, $x \in A$, $y \in B$. Given von Neumann algebras R and S, we let $\mathrm{Bil}^{\sigma}(R, S)$ denote the $F \in \mathrm{Bil}(R, S)$ such that $x \to F(x, y)$ and $y \to F(x, y)$ are normal, i.e., σ-weakly continuous ($x \in R$, $y \in S$). This is a norm closed (S, R) submodule of $\mathrm{Bil}(R, S)$. We define the *binormal projective tensor product* $R \hat{\otimes}^\sigma S$ by

$$R \hat{\otimes}^\sigma S = \mathrm{Bil}^\sigma(R, S)^*.$$

This is a dual (R, S) module.

Given $F \in \mathrm{Bil}^\sigma(R, S)$ and $M \in R \hat{\otimes}^\sigma S$, we use the notation

$$M(F) = \int F\, dM = \int F(x, y)\, dM(x, y).$$

More generally given a dual Banach space V^* and a bounded bilinear function $F: R \times S \to V^*$ such that $x \to F(x, y)(v)$ and $y \to F(x, y)(v)$ are normal for $x, y \in R$, $v \in V$ we define $\int F\, dM \in V^*$ by

$$\left(\int F\, dM\right)(v) = \int F(x, y)(v)\, dM(x, y).$$

We have a natural (R, S) module map

$$\theta: R \otimes S \to R \hat{\otimes}^\sigma S = \mathrm{Bil}^\sigma(R, S)^*$$

defined by letting $\theta(x \otimes y)(F) = F(x, y)$. Since $R_* \otimes S_* \subseteq \mathrm{Bil}^\sigma(R, S)$, and $R_* \otimes S_*$ separates points in $R \otimes S$, θ is one-to-one. We will identify $R \otimes S$ with its image, writing $R \otimes S \subseteq R \hat{\otimes}^\sigma S$. The bilinear map

$$R \times S \to R \otimes S \subseteq R \hat{\otimes}^\sigma S: (x, y) \to x \otimes y$$

is normal in each variable, as are the maps $(x, y) \to rx \otimes y$, $(x, y) \to x \otimes ys$ for $r \in R$, $s \in S$. Using the above conventions, we find

$$M = \int x \otimes y\, dM(x, y)$$

$$rM = \int rx \otimes y\, dM(x, y)$$

$$Ms = \int x \otimes ys\, dM(x, y).$$

A dual (R, S) module V^* is said to be *normal* if the maps $R \to V^*$ defined by $r \to rf$, $s \to fs$ $(r \in R, s \in S, f \in V^*)$ are σ-weak, weak* continuous. The dual V^* of a Banach (R, S) module is generally not normal. In particular since we cannot verify that $R \hat{\otimes}^\sigma S$ is a normal R-bimodule, it is necessary to introduce an auxiliary module $R \hat{\otimes}_0^\sigma S$ below. We say that an element $f \in \text{Bil}(R, S)$ is (internally) *reduced* if there exist states $p \in R_*$, $q \in S_*$ and a constant K such that for all $x \in R$, $y \in S$,

$$|F(x, y)| \leqslant Kp(xx^*)^{1/2} q(y^*y)^{1/2}. \tag{2.1}$$

We let $\text{Bil}_0^\sigma(R, S)$ be the set of all such bilinear functionals. If $\|x_\nu\| \leqslant 1$ and $x_\nu \to 0$ in the strong* topology, then $x_\nu x_\nu^* \to 0$ weakly and $F(x_\nu, y) \to 0$, i.e., F is normal in the first variable. Since the same argument may be applied to the second variable,

$$\text{Bil}_0^\sigma(R, S) \subseteq \text{Bil}^\sigma(R, S).$$

LEMMA 2.1. *Suppose that R, S are arbitrary von Neumann algebras. Then* $\text{Bil}_0^\sigma(R, S)$ *is an (S, R) submodule of* $\text{Bil}^\sigma(R, S)$.

Proof. Given $F_i \in \text{Bil}^\sigma(R, S)$ with

$$|F_i(x, y)| \leqslant K_i p_i(xx^*)^{1/2} q_i(y^*y)^{1/2} \qquad (i = 1, 2),$$

we have that

$$\begin{aligned} |(F_1 + F_2)(x, y)| &\leqslant K_1 p_1(xx^*)^{1/2} q_1(y^*y)^{1/2} + K_2 p_2(xx^*)^{1/2} q_2(y^*y)^{1/2} \\ &\leqslant (K_1 p_1(xx^*) + K_2 p_2(xx^*))^{1/2}(K_1 q_1(y^*y) + K_2 q_2(y^*y))^{1/2} \\ &\leqslant (K_1 + K_2)\, p(xx^*)^{1/2} q(y^*y)^{1/2}, \end{aligned}$$

where if we let $\alpha_i = K_i(K_1 + K_2)^{-1}$, $p = \alpha_1 p_1 + \alpha_2 p_2$, $q = \alpha_1 q_1 + \alpha_2 q_2$. Given $r \in R$ and F satisfying (2.1), we have

$$\begin{aligned} |(Fr)(x, y)| &= |F(rx, y)| \\ &\leqslant Kp(rxx^*r)^{1/2} q(y^*y)^{1/2} \\ &= K_1 p_1(xx^*)^{1/2} q(y^*y)^{1/2}, \end{aligned}$$

where $p_1(u) = p(rur^*)/p(rr^*)$ is a state in R_* and $K_1 = Kp(rr^*)^{1/2}$. A similar calculation applies to the S-operations. Given $\alpha \in \mathbb{C}$, we have $\alpha F = (\alpha 1) F$, hence $\text{Bil}_0^\sigma(R, S)$ is a linear space. ∎

Although it is not evident that it is closed we let $\text{Bil}_0^\sigma(R, S)$ have the *relative norm*, and define the *auxiliary binormal projective tensor product* $R \hat{\otimes}_0^\sigma S$ by

$$R \hat{\otimes}_0^\sigma S = \text{Bil}_0^\sigma(R, S)^*.$$

LEMMA 2.2. *For any von Neumann algebras R and S, $R \hat{\otimes}_0^\sigma S$ is a normal dual Banach (R, S) module.*

Proof. We must show that if $M \in \mathrm{Bil}_0^\sigma(R, S)^*$, then $r \to rM$ and $s \to Ms$ are continuous in the σ-weak, weak* topologies. We may assume that $\|M\| = 1$. Since the inclusion map

$$\theta\colon \mathrm{Bil}_0^\sigma(R, S) \to \mathrm{Bil}(R, S) = (R \mathbin{\hat{\otimes}} S)^*$$

is isometric, the corresponding restriction map

$$\rho = \theta^*\colon (R \mathbin{\hat{\otimes}} S)^{**} \to \mathrm{Bil}_0^\sigma(R, S)^* = R \mathbin{\hat{\otimes}}_0^\sigma S$$

maps the closed unit ball of $(R \mathbin{\hat{\otimes}} S)^{**}$ onto that of $R \mathbin{\hat{\otimes}}_0^\sigma S$. Using the fact that the open unit ball of $R \otimes S$ (with the relative projective norm) is weak* dense in that of $(R \mathbin{\hat{\otimes}} S)^{**}$, we may choose a net

$$M_\nu = \sum_1^{n_\nu} \lambda_k^\nu x_k^\nu \otimes y_k^\nu \in R \otimes S$$

such that $\|x_k^\nu\| = \|y_k^\nu\| = 1$, $0 \leqslant \lambda_k^\nu$, $\sum_k \lambda_k^\nu = 1$, and

$$M(F) = \lim_\nu \sum_{k=1}^{n_\nu} \lambda_k^\nu F(x_k^\nu, y_k^\nu)$$

for each $F \in \mathrm{Bil}_0^\sigma(R, S)$.

Fixing $F \in \mathrm{Bil}_0^\sigma(R, S)$, we must show that the functions $r \to (rM)(F)$ and $s \to (Ms)(F)$ are σ-weakly continuous. Choosing states $p \in R_*$, $q \in S_*$ and $K > 0$ such that

$$|F(x, y)| \leqslant Kp(xx^*)^{1/2}\, q(y^*y)^{1/2},$$

we have for all ν, $rM_\nu(F) = M_\nu(Fr)$, and thus

$$\begin{aligned}
|rM_\nu(F)| &= \left| \sum_{k=1}^{n_\nu} \lambda_k^\nu F(rx_k^\nu, y_k^\nu) \right| \\
&\leqslant K \sum_{k=1}^{n_\nu} \lambda_k^\nu\, p(rx_k^\nu x_k^{\nu *} r^*)^{1/2}\, q(y_k^{\nu *} y_k^\nu)^{1/2} \\
&\leqslant K \sum_{k=1}^{n_\nu} \lambda_k^\nu\, p(rr^*)^{1/2} \\
&= Kp(rr^*)^{1/2},
\end{aligned}$$

i.e.,

$$|rM(F)| \leqslant Kp(rr^*)^{1/2}.$$

Given a net $r_\alpha \in R$ with $\|r_\alpha\| \leqslant 1$ and $r_\alpha \to 0$ σ-strongly, it follows that

$r_\alpha M(F) \to 0$, and thus $r \to rM(F)$ is σ-weakly continuous. A similar argument applies to the function $s \to (Ms)(F)$. ▮

As was the case for $R \hat{\otimes}^\sigma S$, there is an obvious map

$$\theta_0: R \otimes S \to R \hat{\otimes}_0^\sigma S = \mathrm{Bil}_0^\sigma(R, S)^*.$$

This is again an injection because we have

$$R_* \otimes S_* \subseteq \mathrm{Bil}_0^\sigma(R, S).$$

To see the latter we may suppose that $f \in R_*$, $g \in S_*$ satisfy $\|f\| = \|g\| = 1$, and have polar decompositions.

$$f(r) = p(vr), \qquad g(s) = q(ws),$$

where p, q are states and v, w are partial isometries. Then

$$\begin{aligned} |(f^* \otimes g)(r \otimes s)| &= |p(rv^*)|\, |q(ws)| \\ &\leqslant p(rr^*)^{1/2}\, p(vv^*)^{1/2}\, q(ww^*)^{1/2}\, q(s^*s)^{1/2} \\ &\leqslant p(rr^*)^{1/2}\, q(s^*s)^{1/2}. \end{aligned}$$

We shall identity $R \otimes S$ with an (R, S) submodule of $R \hat{\otimes}_0^\sigma S$.

Letting $R = S$, we claim that the multiplication map

$$\pi: R \otimes R \to R: x \otimes y \to xy$$

has a unique weak* continuous extension to $R \hat{\otimes}_0^\sigma R$. Given $f \in R_*$ with $\|f\| = 1$, we let $f(x) = p(vx)$ be the polar decomposition as above. Then we have

$$\begin{aligned} |f(xy)| = |p(vxy)| &\leqslant p(vxx^*v^*)^{1/2}\, p(y^*y)^{1/2} \\ &= Kp_1(xx^*)^{1/2}\, p(y^*y)^{1/2}, \end{aligned}$$

where $p_1(y) = p(vyv^*)/p(vv^*)$ is a state in R_* and $K = p(vv^*)^{1/2}$ and thus $f \circ \pi \in \mathrm{Bil}_0^\sigma(R, R)$. We define

$$\pi_0: R_* \to \mathrm{Bil}_0^\sigma(R, R): f \to f \circ \pi,$$

and we let

$$\bar{\pi} = \pi_0^*: R \hat{\otimes}_0^\sigma R \to R.$$

It is a simple matter to verify that $\bar{\pi}|_{R \otimes R} = \pi$. Also, $R \otimes R$ is weak* dense in $R \hat{\otimes}_0^\sigma R$ because if $F \in \mathrm{Bil}_0^\sigma(R, R)$ and $F|_{R \otimes R} = 0$, then $F = 0$. Since $\bar{\pi}$ is weak* continuous, it is the unique such extension of π. We will generally write π rather than $\bar{\pi}$. It should be noted that a simpler version of this argument similarly shows that π also has a unique weak* extension

$$\pi: R \hat{\otimes}^{\sigma} R \to R.$$

The following will play a key role in Section 3.

LEMMA 2.3. *Suppose that R is a finite or properly infinite von Neumann algebra. Then there is a weak* continuous linear R bimodule map*

$$\Phi: R \hat{\otimes}_0^{\sigma} R \to R \hat{\otimes}^{\sigma} R$$

such that $\pi \circ \Phi = \pi$.

Proof. Let us first assume that R is finite. We let $\mathscr{U}(R)$ denote the unitary group in R, and $\Gamma(R)$ be all non-negative functions f on $\mathscr{U}(R)$ which are zero at all but finitely many unitaries and satisfy $\sum f(u) = 1$. Given $f \in \Gamma(R)$, we define $\phi_f: R \to R$ by letting

$$\phi_f(r) = \sum f(u)\, uru^*.$$

Given $r \in R$, we let $\kappa(r)$ be the norm closed convex hull of the operators uru^*, $u \in \mathscr{U}(R)$, and $Z(r)$ be the unique element in $\kappa(r) \cap Z(R)$. For each finite set $\Delta = \{r_1, ..., r_p\} \subseteq R$ and $n \in \mathbb{N}$, we may choose an $f = f_{\Delta}, \in \mathscr{F}(R)$ and central elements $z_i \in Z(R)$ with $\|\phi_f(r_i) - z_i\| < 1/n$ [4, III. Sect. 5, Lemma 4]. Since $Z(\phi_f(r_i)) = Z(z_i) = z_i$, it follows that, $\|Z(r_i) - z_i\| < 1/n$, and thus $\|\phi_f(r_i) - Z(r_i)\| < 2/n$. The set N of indices $v = (\Delta, n)$ is directed under the partial order $(\Delta, n) \leqslant (\Delta', n')$ if $\Delta \subseteq \Delta'$ and $n \leqslant n'$, and we see that for $r \in R$,

$$\lim_{v} \phi_v(r) = Z(r),$$

in the norm topology, where we have let $\phi_v = \phi_{f(\Delta,n)}$ for $v = (\Delta, n)$.

We fix a character ω in the spectrum of $l^{\infty}(N)$ which is a limit point of the net (Δ, n) (as usual, points in N are also characters), and if $h \in l^{\infty}(N)$, we let

$$\lim_{v \to \omega} h(v) = h(\omega).$$

In particular, if the net $h(v)$ converges in the usual sense, we have

$$\lim_{v \to \omega} h(v) = \lim_{v} h(v).$$

Given $F \in \mathrm{Bil}^\sigma(R, R)$ and $x, y \in R$, we have that

$$h: \nu \to \theta_\nu(F)(x, y) = \sum f_\nu(u)\, F(xu, u^*y)$$

is bounded since

$$|\theta_\nu(F)(x, y)| \leqslant \sum f_\nu(u)\, \|F\|\, \|x\|\, \|y\| = \|F\|\, \|x\|\, \|y\|, \tag{2.2}$$

hence we may define $\theta(F) \in \mathrm{Bil}(R, R)$ by

$$\theta(F)(x, y) = \lim_{\nu \to \omega} \theta_\nu(F)(x, y).$$

From (2.2) it follows that $\|h\|_\infty \leqslant \|F\|\, \|x\|\, \|y\|$, and thus

$$|\theta(F)(x, y)| = |h(\omega)| \leqslant \|F\|\, \|x\|\, \|y\|,$$

i.e., $\|\theta(F)\| \leqslant \|F\|$.

Fixing $F \in \mathrm{Bil}^\sigma(R, R)$, we have from one of Haagerup's extensions of the Grothendieck–Pisier Inequality [6, Proposition 2.3] that there exist states $p_1, p_2, q_1, q_2 \in R_*$ such that

$$|F(x, y)| \leqslant \|F\| (p_1(x^*x) + p_2(xx^*))^{1/2} (q_1(y^*y) + q_2(yy^*))^{1/2}.$$

Given $\nu \in N$, it follows that

$$\begin{aligned}
|\theta_\nu(F)(x, y)| &= \left| \sum f_\nu(u)\, F(xu^*, uy) \right| \\
&\leqslant \|F\| \sum f_\nu(u) (p_1(ux^*xu^*) + p_2(xx^*))^{1/2} \\
&\quad \times (q_1(y^*y) + q_2(uyy^*u^*))^{1/2} \\
&\leqslant \|F\| \left[\sum f_\nu(u)\, p_1(ux^*xu^*) + p_2(xx^*) \right]^{1/2} \\
&\quad \times \left[q_1(y^*y) + \sum f_\nu(u)\, q_2(uyy^*u^*) \right]^{1/2} \\
&= \|F\| [p_1(\phi_\nu(x^*x)) + p_2(xx^*)]^{1/2} \\
&\quad \times [q_1(y^*y) + q_2(\phi_\nu(yy^*))]^{1/2}.
\end{aligned}$$

Letting $\sigma_1 = p_1 \circ Z$, $\sigma_2 = q_2 \circ Z$, it follows that σ_i are traces such that

$$\begin{aligned}
|\theta(F)(x, y)| &\leqslant \|F\| [\sigma_1(x^*x) + p_2(xx^*)]^{1/2} [q_2(y^*y) + \sigma_2(yy^*)]^{1/2} \\
&= 2\, \|F\|\, p(xx^*)^{1/2}\, q(y^*y)^{1/2},
\end{aligned}$$

where $p = \frac{1}{2}(\sigma_1 + p_2)$ and $q = \frac{1}{2}(q_1 + \sigma_2)$. We conclude that $\theta(F) \in \mathrm{Bil}_0^\sigma(R, R)$.

We have that

$$\theta\colon \mathrm{Bil}^{\sigma}(R, R) \to \mathrm{Bil}_0^{\sigma}(R, R)$$

is trivially a bimodule map, and thus the same is true for

$$\Phi = \theta^*\colon R \mathbin{\hat{\otimes}}_0^{\sigma} R \to R \mathbin{\hat{\otimes}}^{\sigma} R.$$

Furthermore, we have for $F \in \mathrm{Bil}^{\sigma}(R, R)$

$$\begin{aligned}\left\langle \Phi\left(\sum_{i=1}^{n} x_i \otimes y_i\right), F\right\rangle &= \left\langle \sum_{i=1}^{n} x_i \otimes y_i, \theta(F)\right\rangle \\ &= \lim_{\nu \to \omega} \sum_{i=1}^{n} \theta_{\nu}(F)(x_i, y_i) \\ &= \lim_{\nu \to \omega} \sum_{i=1}^{n} \sum_{u} f_{\nu}(u)\, F(x_i u^*, u y_i) \\ &= \lim_{\nu \to \omega} \left\langle \sum_{i,u} f_{\nu}(u)\, x_i u^* \otimes u y_i, F\right\rangle,\end{aligned}$$

and thus $\Phi(\sum_{i=1}^{n} x_i \otimes y_i)$ is the weak* limit of the net

$$z_{\nu} = \sum_{i,u} f_{\nu}(u)\, x_i u^* \otimes u y_i.$$

It follows that

$$\begin{aligned}\pi \circ \Phi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) &= \lim_{\nu \to \omega} \pi(z_{\nu}) \\ &= \lim_{\nu \to \omega} \sum_{i,u} f_{\nu}(u)\, x_i y_i \\ &= \pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right),\end{aligned}$$

and since $R \otimes R$ is dense in both $R \otimes_0^{\sigma} R$ and $R \otimes^{\sigma} R$, $\pi \circ \Phi = \pi$.

If R is properly infinite, we have from [7, Proof of Lemma 2] that there exists a sequence $v_n \in R$ such that $v_n^* v_n = 1$ and $v_n v_n^* \to 0$ weakly. Letting ω be a limit point of the sequence 1, 2, ... in the spectrum of $l^{\infty}(\mathbf{N})$, we define

$$\theta\colon \mathrm{Bil}^{\sigma}(R, R) \to \mathrm{Bil}(R, R)$$

by

$$\theta(F)(x, y) = \lim_{n \to \omega} F(x v_n^*, v_n y)$$

(see above). We claim that

$$\theta(F) \in \mathrm{Bil}_0^\sigma(R, R) \subseteq \mathrm{Bil}^\sigma(R, R).$$

To see this, suppose that $\|x\|, \|y\| \leqslant 1$, and choosing $p_i, q_i \in R_*$ as above,

$$\begin{aligned}|F(xv_n^*, v_n y)| &\leqslant \|F\| [p_1(v_n x^* x v_n^*) + p_2(xx^*)]^{1/2} [q_1(y^*y) + q_2(v_n yy^* v_n^*]^{1/2} \\ &\leqslant \|F\| [p_1(v_n v_n^*) + p_2(xx^*)]^{1/2} [q_1(y^*y) + q_2(v_n v_n^*)]^{1/2}\end{aligned}$$

since $0 \leqslant x^*x \leqslant 1$ and $0 \leqslant yy^* \leqslant 1$. Since $p_1(v_n v_n^*)$, $q_2(v_n v_n^*) \to 0$, we conclude that

$$|\theta(F)(x, y)| \leqslant p_2(xx^*)^{1/2} q_1(y^*y)^{1/2}.$$

The remainder of the argument coincides with that given for the finite case.

3. Amenability

Given a unital Banach algebra A and a normed A bimodule V, a *derivation* $\delta: A \to V$ is a bounded linear map $\delta: A \to V$ such that

$$\delta(ab) = \delta(a)\, b + a\delta(b).$$

We say that δ is *inner* if there exists a $v_0 \in V$ such that

$$\delta(a) = av_0 - v_0 a.$$

A unital Banach algebra A is said to be *amenable* if for any dual A bimodule V^*, any derivation $\delta: A \to V^*$ is inner. A von Neumann algebra R is said to be *amenable* (*in the von Neumann algebra sense*) if for any dual normal R bimodule, any normal derivation $\delta: R \to V^*$ is inner.

We note that in the usual definition of normal cohomology for von Neumann algebras (see, e.g., [9, 10, 11]), one considers only the duals V^* of *Banach* bimodules V. However, given a normed module V with completion $\bar{V}$, the module operations extend uniquely to $\bar{V}$. If $f \in V^*$, f has a unique extension $\bar{f} \in (\bar{V})^*$. The map

$$\theta: V^* \to (\bar{V})^*: f \to \bar{f}$$

is clearly an isometric R-bimodule isomorphism. Furthermore, V^* is a normal dual bimodule if and only if that is the case for $(\bar{V})^*$. To see this, suppose that V^* is normal (the converse is trivial). Then given $\bar{v} \in \bar{V}$, and $\bar{f} \in \bar{V}^*$, let $v_n \in V$ be such that $\|v_n - \bar{v}\| \to 0$. Then the functions $r \to (rf)(v_n)$

converge uniformly to the functions $r \to (r\bar{f})(\bar{v})$ on the closed unit ball R_1, since

$$|(r\bar{f})(\bar{v}) - (r\bar{f})(v_n)| \leqslant \|f\| \, \|v_n - \bar{v}\|.$$

It follows that $r \to (rf)(v)$ is weakly continuous on R_1, and thus σ-weakly continuous.

If R is amenable with respect to normal bimodules V^*, V a Banach space, then it is also amenable in our sense. This follows since given a normed vector space V and a normal derivation $\delta: R \to V^*$ where V^* is a normal bimodule, the above uniform convergence argument shows that $\delta: R \to (\bar{V})^*$ is also normal. It follows that $\delta(r) = rf_0 - f_0 r$ for some $f_0 \in (\bar{V})^* \cong V^*$.

Given a von Neumann algebra R, a *normal virtual diagonal* for R is an element $M \in R \,\hat{\otimes}^\sigma R$ such that $rM = Mr$ for all $r \in R$, and $\pi(M) = 1$. Letting $M = \int x \otimes y \, dM(x, y)$, these relations may be rewritten

$$\int rx \otimes y \, dM(x, y) = \int x \otimes yr \, dM(x, y) \tag{3.1}$$

$$\int xy \, dM(x, y) = 1. \tag{3.2}$$

Note that for the latter we are using the fact that $(x, y) \to xy$ has range in the dual Banach space $R = (R_*)^*$. If $M \in R \,\hat{\otimes}_0^\sigma R$ has the above properties, we call it a *reduced* normal virtual diagonal. The "classic" example of a "diagonal" in $R = M(n)$ is

$$M = \sum e_{i1} \otimes e_{1i} \in R \otimes R,$$

where e_{ij} are the matrix units (see [1, Chap. IX, Proposition 7.7]).

THEOREM 3.1. *A von Neumann algebra R is amenable if and only if it has a normal virtual diagonal M.*

Proof. Let us suppose that $M \in R \,\hat{\otimes}^\sigma R$ is a normal virtual diagonal. Given a normal derivation $\delta: R \to V^*$, the bilinear function $F(x, y) = \delta(x)\, y$ is normal in each variable. Thus we may let

$$f_0 = \int F(x, y) \, dM(x, y) = \int \delta(x)\, y \, dM(x, y) \in V^*.$$

We have for $r \in R$,

$$\begin{aligned} rf_0 &= \int r\delta(x)\, y\, dM(x, y) \\ &= \int \delta(rx)\, y\, dM(x, y) - \int \delta(r)\, xy\, dM(x, y) \\ &= \int \delta(x)\, yr\, dM(x, y) - \delta(r) \int xy\, dm(x, y) \\ &= f_0 r - \delta(r), \end{aligned}$$

where in the third step we used

$$\int F(rx, y)\, dM(x, y) = \left\langle \int rx \otimes y\, dM(x, y), F \right\rangle$$
$$= \int \langle x \otimes yr\, dM(x, y), F \rangle = \int F(x, yr)\, dM(x, y).$$

Conversely if R is amenable, consider the normal dual R bimodule

$$R \mathbin{\hat{\otimes}}_0^\sigma R = V^*, \qquad V = \mathrm{Bil}_0^\sigma(R, R).$$

We have that

$$W = \ker \pi \subseteq R \mathbin{\hat{\otimes}}_0^\sigma R$$

is weak* closed since $\pi = \pi_0^*$ is weak* continuous. Letting

$$W_\perp = \{v \in V: f(v) = 0 \text{ for } \textit{all } f \in W\},$$

it follows that $W \cong (V/W_\perp)^*$ (see [3, Chap. II, Sect. 1, Lemma 1]), and since π is an R-bimodule map, we see that W is itself a normal dual R-bimodule. Letting $M_1 = 1 \otimes 1 \in R \mathbin{\hat{\otimes}}_0^\sigma R$, the map

$$\delta_1: R \to R \mathbin{\hat{\otimes}}_0^\sigma R: r \to M_1 r - r M_1$$

may also be regarded as a derivation into W since

$$\pi(M_1 r - r M_1) = \pi(1 \otimes r - r \otimes 1) = 0.$$

Thus since W is a normal dual R bimodule, there is an $M_0 \in W$ such that

$$M_1 r - r M_1 = M_0 r - r M_0.$$

Letting $M = M_1 - M_0$, it follows that $rM = Mr$ for all $r \in R$, and

$$\pi(M) = \pi(M_1) - \pi(M_0) = 1.$$

If R is finite or properly infinite, we have from Lemma 2.3 that $\tilde{M} = \Phi(M) \in R \hat{\otimes}^\sigma R$ is a normal virtual diagonal because

$$r\tilde{M} = \Phi(rM) = \Phi(Mr) = \tilde{M}r,$$
$$\pi(\tilde{M}) = \pi\Phi(M) = \pi(M) = 1.$$

In the general case, let e_1 and e_2 be the unique central projections with $1 = e_1 + e_2$, and R_{e_1} finite, R_{e_2} properly infinite. We may select normal virtual diagonals

$$M_i \in R_{e_i} \hat{\otimes}^\sigma R_{e_i}.$$

Given $F \in \mathrm{Bil}^\sigma(R, R)$, we have

$$F_i(r, s) = F(re_i, se_i) \qquad (r, s \in R_{e_i})$$

defines an element of $\mathrm{Bil}^\sigma(R_{e_i}, R_{e_i})$, and we may define $M \in R \hat{\otimes}^\sigma R$ by

$$\langle M, F \rangle = \langle M_1, F_1 \rangle + \langle M_2, F_2 \rangle.$$

We have for $r \in R$, $s, t \in R_{e_i}$

$$(Fr)_i(s, t) = (Fr)(se_i, te_i) = (F_i r_i)(s, t),$$

where $r_i = r|_{e_i H} \in R_{e_i}$, and thus

$$\begin{aligned} \langle rM, F \rangle &= \langle M, Fr \rangle \\ &= \langle M_1, (Fr)_1 \rangle + \langle M_2, (Fr)_2 \rangle \\ &= \langle r_1 M_1, F_1 \rangle + \langle r_2 M_2, F_2 \rangle \\ &= \langle M_1 r_1, F_1 \rangle + \langle M_2 r_2, F_2 \rangle \\ &= \langle Mr, F \rangle, \end{aligned}$$

where we have used symmetry to obtain the last equality.

Given $f \in R_*$, we define $f_i \in (R_{e_i})_*$ by $f_i(r) = f(re_i)$. Then we have for $r, s \in R_{e_i}$,

$$\begin{aligned} \pi_0(f_i)(r, s) &= f_i(rs) \\ &= f(rse_i) \\ &= \pi_0(f)(re_i, se_i) \\ &= \pi_0(f)_i(r, s), \end{aligned}$$

and thus letting 1_i be the identity of R_{e_i},

$$\begin{aligned}\langle \pi(M), f\rangle &= \langle M, \pi_0(f)\rangle \\ &= \langle M_1, \pi_0(f)_1\rangle + \langle M_2, \pi_0(f)_2\rangle \\ &= \langle \pi(M_1), f_1\rangle + \langle \pi(M_2), f_2\rangle \\ &= \langle 1_1, f_1\rangle + \langle 1_2, f_2\rangle \\ &= f(e_1) + f(e_2) \\ &= \langle 1, f\rangle,\end{aligned}$$

and $\pi(M) = 1$. ∎

COROLLARY 3.2 [7]. *All nuclear C^*-algebras are amenable.*

Proof. If A is a nuclear C^*-algebra, then from [2], $R = A^{**}$ is an amenable von Neumann algebra. We let $M \in R \hat{\otimes}^\sigma R$ be a normal virtual diagonal for A^*. Given $F \in \mathrm{Bil}(A, A)$ we have that F has a unique extension to an element $\tilde{F} \in \mathrm{Bil}^\sigma(R, R)$ (see [9, Lemma 2.1]). We define $M_0 \in \mathrm{Bil}(A, A)^*$ by $M_0(F) = M(\tilde{F})$. Then it is easy to check that M_0 is a virtual diagonal for A, i.e., $aM_0 = M_0 a$ and $\pi(M_0) = 1$, where

$$\pi\colon \mathrm{Bil}(A, A)^* \to A$$

is the unique weak* continuous extension of multiplication. Thus from Johnson's original result [8], A is amenable.

4. SOME CONCLUDING REMARKS

Normal virtual diagonals may be used to clarify Connes' original proof that if R is an amenable von Neumann algebra, then R is injective. For the case of a II_1 von Neumann algebra R, Connes' problem reduced to showing that a trace τ on R has a central extension, i.e., a "hypertrace" $\omega \in B(H)^*$. Putting R in "standard position," we may assume that $\tau(r) = rx_0 \cdot x_0$ for a unit vector $x_0 \in H$. Then $\omega_0(b) = bx_0 \cdot x_0$ is an extension of τ to a functional $\omega_0 \in B(H)^*$. In order to apply amenability directly, it was necessary to replace $B(H)^*$ by a suitable normal dual R bimodule (see [12] for another approach).

Using a normal virtual diagonal, we may instead define $\omega \in B(H)^*$ directly by letting

$$\omega(b) = \int \omega_0(ybx)\, dM(x, y).$$

Here we are using the fact that $\phi\colon (x, y) \to \omega_0(ybx)$ is an element of $B^\sigma(R, R)$. It follows that for any $r \in R$,

$$\begin{aligned}
\omega(rb) &= \int \omega_0((yr)\,bx)\,dM(x, y) \\
&= \int \phi(x, yr)\,dM(x, y) \\
&= \int \phi(rx, y)\,dM(x, y) \\
&= \int \omega_0(ybrx)\,dM(x, y) \\
&= \omega(br).
\end{aligned}$$

On the other hand, $\omega|_R = \tau$ since if $r \in R$,

$$\begin{aligned}
\omega(r) &= \int \omega_0(yrx)\,dM(x, y) \\
&= \int \tau(yrx)\,dM(x, y) \\
&= \int \tau(rxy)\,dM(x, y) \\
&= \left\langle r\tau, \int xy\,dM(x, y)\right\rangle \\
&= \langle r\tau, 1\rangle \\
&= \tau(r).
\end{aligned}$$

In the particular case where $R = R(G)$ is the regular group von Neumann algebra of an amenable countable discrete group G, one may use the auxiliary bimodule to construct a normal virtual diagonal. A *mean* on G is an element $m \in l^\infty(G)^*$ such that $m \geqslant 0$ and $m(1) = 1$. Given such an m and a bounded function $\phi\colon G \to V^*$ where V^* is a dual Banach space, we define $\int \phi(s)\,dm(s) \in V^*$ by

$$\int \phi(s)\,dm(s)(v) = \int \phi(s)\,v\,dm(s).$$

We have that $l^\infty(G)$, and thus $l^\infty(G)^*$, is a G bimodule in the usual way, and we say that a mean m is *invariant* if $sm = m = ms$. G is called *amenable* if it has an invariant mean m. If that is the case, consider the map

$$\phi: G \to R \hat{\otimes}_0^\sigma R: s \to \lambda(s)^* \otimes \lambda(s),$$

where $\lambda(s): l^2(G) \to l^2(G)$ is a left translation (recall that $R(G)$ is generated by these operators). We claim that

$$M = \int \lambda(s)^* \otimes \lambda(s)\, dm(s) \in R \hat{\otimes}_0^\sigma R$$

is a reduced normal virtual diagonal. Given $t \in G$, it is evident that $\lambda(t) M = M\lambda(t)$. It follows that $rM = Mr$ for a σ-weakly dense set of $r \in R$. Since $R \hat{\otimes}_0^\sigma R$ is a normal module, this is true for all $r \in R$. On the other hand, if $f \in R_*$,

$$\begin{aligned}
\langle \pi(M), f \rangle &= \langle M, \pi_0(f) \rangle \\
&= \int \langle \lambda(s)^* \otimes \lambda(s), \pi_0(f) \rangle\, dm(s) \\
&= \int \langle 1, f \rangle\, dm(s) \\
&= \langle 1, f \rangle,
\end{aligned}$$

and $\pi(M) = 1$. From Lemma 2.3, $\Phi(M) \in R \hat{\otimes}^\sigma R$ is a normal virtual diagonal.

It is tempting to use the above approach to study the "Inner Amenability Problem." Specifically, it is not known whether inner amenability for G implies that $R(G)$ has property Γ (the converse was proved in [5]—see also [13]). A mean $m \in l^\infty(G)^*$ is said to be *inner invariant* if for any $h \in l^\infty(G)$

$$\int h(t^{-1}st)\, dm(s) = \int h(s)\, dm(s),$$

and *non-trivial* if in addition, $m(\delta_1) < 1$. G is said to be *inner amenable* if it has such a mean. We then have

$$M = \int \lambda(s)^* \otimes \lambda(s)\, dm(s) \in R \hat{\otimes}_0^\sigma R$$

satisfies

$$\lambda(t) \otimes \lambda(t)\, M = M\lambda(t) \otimes \lambda(t),$$

where we are using the "internal" as well as the "external" module operations. Unfortunately, $R \hat{\otimes}_0^\sigma R$ is probably not a normal module under these operations; hence we cannot conclude that $(r \otimes r) M = M(r \otimes r)$ for all $r \in R$. Furthermore, it should be noted that the map Φ of Lemma 2.3 does not preserve these operations.

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Note added in proof. The reduced bilinear functionals are just those that are *completely bounded* in the sense of Christensen and Sinclair (see [14], [15]).

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